

On level crossing in conformal field theories

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ABSTRACT: We study the properties of operators in a unitary conformal field theory whose scaling dimensions approach each other for some values of the parameters and satisfy von Neumann-Wigner non-crossing rule. We argue that the scaling dimensions of such operators and their OPE coefficients have a universal scaling behavior in the vicinity of the crossing point. We demonstrate that the obtained relations are in a good agreement with the known examples of the level-crossing phenomenon in maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory, three-dimensional conformal field theories and QCD.

KEYWORDS: 1/N Expansion, Conformal and W Symmetry, Field Theories in Lower Dimensions, Supersymmetric gauge theory

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1 Introduction

Let us consider a unitary conformal field theory (CFT) and examine the spectrum of scaling dimensions of conformal primary operators. The scaling dimensions Δ_i depend on various parameters of CFT such as coupling constants, Casimirs of internal symmetry group etc. In a close analogy with gauge theories, we assume that CFT depends on a small parameter $1/N$ and the scaling dimensions admit an expansion in powers of $1/N^2$, e.g.

$$\Delta_i = \Delta_i^{(0)} + \frac{1}{N^2}\Delta_i^{(1)} + \frac{1}{N^4}\Delta_i^{(2)} + \dots \quad (1.1)$$

Here the leading term $\Delta_i^{(0)}$ is an eigenvalue of the dilatation operator in the planar limit, the subleading terms can be obtained by diagonalizing the dilatation operator perturbatively in powers of $1/N$.

In this note we ask the question what happens when the scaling dimensions of two operators collide in the planar limit for some value of the coupling constant, $\Delta_1^{(0)} = \Delta_2^{(0)}$. This occurs in particular in maximally supersymmetric $\mathcal{N} = 4$ super Yang-Mills theory (SYM) for Konishi and double-trace operators (defined in (4.1) below). The scaling dimension of the double-trace operator is protected from corrections in the planar limit, whereas the scaling dimension of the Konishi operator is an increasing function of 't Hooft coupling constant that starts at $2 + O(\lambda)$ at weak coupling and increases as $2\lambda^{1/4}$ at strong coupling [1]. As a consequence, the two levels cross each other at finite λ . We shall return to this example below.

It is well-known that, in the radial quantization, the dilatation operator in D -dimensional Euclidean CFT coincides with the Hamiltonian generating time translations on the cylinder $\mathbb{R} \times S^{D-1}$. In a unitary CFT, it is a hermitian operator with respect to the scalar

product given by two-point correlation function of $O_i(x)$ on the cylinder and, therefore, its eigenvalues Δ_i have to satisfy von Neumann-Wigner non-crossing rule [2] stating that the levels of the dilatation operator with the same symmetry cannot cross.

How can we reconcile the non-crossing rule with the fact that the functions $\Delta_i^{(0)}$ providing the leading (planar) correction to (1.1) can cross each other? The situation here is very similar to that in quantum mechanics — the perturbative expansion (1.1) is well-defined if the energy level separation $\Delta_1 - \Delta_2$ is much larger compared to their ‘interaction energy’ defined by the leading nonplanar $O(1/N)$ correction to the dilatation operator. In the opposite limit, for

$$\epsilon \equiv \Delta_1 - \Delta_2 = O(1/N), \quad (1.2)$$

the perturbative expansion (1.1) becomes singular due to small denominators and needs to be resummed to all orders in $1/N$. As we show below, the resummed expressions for Δ_i indeed respect the non-crossing rule.

2 Resummed scaling dimensions

Let us consider a pair of conformal operators \mathcal{O}_1 and \mathcal{O}_2 whose scaling dimensions depend on the coupling constant g and admit the $1/N$ expansion (1.1). For the sake of simplicity we take $\mathcal{O}_i(x)$ to be scalar operators normalized as

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(0) \rangle = \frac{\delta_{ij}}{(x^2)^{\Delta_i}}. \quad (2.1)$$

We assume that for $N \rightarrow \infty$ their scaling dimensions intersect at some value of the coupling constant, $\Delta_1^{(0)}(g) = \Delta_2^{(0)}(g)$ for $g = g_*$, and remain separated from the rest of the spectrum of Δ_i ’s by a gap that remains finite for $N \rightarrow \infty$.

We tacitly assume throughout this section that the scaling dimensions of the operators \mathcal{O}_1 and \mathcal{O}_2 satisfy (1.2). This translates into the following condition for the coupling constant $|g - g_*| = O(1/N)$. We show below that for N large but finite, the scaling dimensions Δ_1 and Δ_2 do not intersect as one varies the coupling constant in the vicinity of $g = g_*$, the minimal distance between the two curves scales as $O(1/N)$.¹

Let O_1 and O_2 be the eigenstates of the dilatation operator in the planar limit, so that $\mathcal{O}_i(x) = O_i(x) + O(1/N)$. The conformal symmetry implies that away from the crossing point, for $\Delta_1^{(0)} \neq \Delta_2^{(0)}$, the two-point correlation function $\langle O_1(x) O_2(0) \rangle$ should vanish for $N \rightarrow \infty$. This does not mean however that $\langle O_1(x) O_2(0) \rangle$ can not be different from zero. For N large but finite it can receive corrections suppressed by powers of $1/N$

$$\langle O_1(x) O_2(0) \rangle = \frac{1}{N} \varphi(x^2) + \dots, \quad (2.2)$$

where dots denote subleading corrections. In the similar manner, the leading nonplanar corrections to the diagonal correlation functions $\langle O_i(x) O_i(0) \rangle$ scale as $O(1/N^2)$.

The function $\varphi(x^2)$ depends on the coupling constant and describes the mixing between the operators O_1 and O_2 . If the scaling dimensions of the operators O_1 and O_2

¹A similar result was independently obtained by Fernando Alday (private communication).

satisfy (1.2) and are separated from the rest of the spectrum by a finite gap, we expect that the conformal operators are given by their linear combinations

$$O_+ = O_1 + c_2 O_2, \quad O_- = O_2 + c_1 O_1. \quad (2.3)$$

Requiring $\langle O_+(x)O_-(0) \rangle = 0$ we find with a help of (2.3), (2.1) and (2.2) that, in the vicinity of the crossing point, the conformal symmetry restricts the possible form of the function $\varphi(x^2)$

$$\begin{aligned} \frac{1}{N} \varphi(x^2) (1 + c_1 c_2) &= -c_1 (x^2)^{-\Delta_1} - c_2 (x^2)^{-\Delta_2} \\ &= - (x^2)^{-\Delta_1} [c_1 + c_2 + c_2 \epsilon \ln x^2]. \end{aligned} \quad (2.4)$$

Here in the second relation we took into account (1.2) and neglected terms proportional to $\epsilon^2 \sim 1/N^2$. Then, going to $N \rightarrow \infty$ limit on the both sides of (2.4) and taking into account that $\epsilon = O(1/N)$, we find $c_2 = -c_1 + O(1/N)$ together with

$$\varphi(x^2) = \frac{\gamma \ln x^2}{(x^2)^{\Delta_1}}, \quad (2.5)$$

where γ does not depend on N and satisfies

$$c_1^2 + c_1 \frac{\epsilon N}{\gamma} - 1 = 0. \quad (2.6)$$

We can use this equation to express the coefficients c_1 and c_2 in terms of γ and obtain from (2.3) the conformal operators O_{\pm} that take into account the leading nonplanar correction.²

To determine the scaling dimensions of the operators (2.3), we examine their two-point correlation functions

$$\begin{aligned} \langle O_-(x)O_-(0) \rangle &= \frac{1}{(x^2)^{\Delta_2}} + \frac{c_1^2}{(x^2)^{\Delta_1}} + \frac{2c_1/N}{(x^2)^{\Delta_1}} \gamma \ln x^2 \\ &= \frac{1 + c_1^2}{(x^2)^{\Delta_2}} \left[1 + \frac{c_1(2\gamma/N - c_1 \epsilon)}{1 + c_1^2} \ln x^2 \right] \end{aligned} \quad (2.7)$$

and similar for $\langle O_+(x)O_+(0) \rangle$. Matching this relation into (2.1) we obtain the leading $O(1/N)$ correction to the scaling dimension of the operator O_-

$$\Delta_- = \Delta_2 - \frac{c_1(2\gamma/N - c_1 \epsilon)}{1 + c_1^2} = \Delta_2 - \frac{\gamma}{N} c_1, \quad (2.8)$$

where in the last relation we applied (2.6). Going through similar calculation of Δ_+ we find

$$\Delta_{\pm} = \frac{\Delta_1 + \Delta_2}{2} \pm \sqrt{\frac{\epsilon^2}{4} + \frac{\gamma^2}{N^2}}, \quad (2.9)$$

with $\epsilon = \Delta_1 - \Delta_2$. We recall that this relation holds up to corrections suppressed by powers of $1/N$.

²The two solutions to (2.6) give rise to the same operators O_{\pm} (up to an overall normalization factor).

The following comments are in order.

An attentive reader will likely notice that Δ_{\pm} coincide with energies of a two-level system with a Hamiltonian

$$\mathbb{H} = \begin{bmatrix} \Delta_1 & \gamma/N \\ \gamma/N & \Delta_2 \end{bmatrix}. \quad (2.10)$$

Indeed, we can use (2.1), (2.2) and (2.5) to verify that this matrix defines the action of the dilatation operator on the operators O_1 and O_2 , that is $i[\mathbb{D}, O_i(0)] = \mathbb{H}_{ij} O_j(0)$. Then, the construction of the conformal operators (2.3) follows the usual consideration of the operator mixing in gauge theory at one loop with the only difference that $1/N$ plays the role of 't Hooft coupling constant.³

It is instructive to compare (2.9) with the general expression (1.1). Expanding Δ_{\pm} in powers of $1/N$ we find from (2.9)

$$\begin{aligned} \Delta_+ &= \Delta_1 + \frac{\gamma^2/\epsilon}{N^2} - \frac{\gamma^4/\epsilon^3}{N^4} + \dots, \\ \Delta_- &= \Delta_2 - \frac{\gamma^2/\epsilon}{N^2} + \frac{\gamma^4/\epsilon^3}{N^4} + \dots. \end{aligned} \quad (2.11)$$

Comparison with (1.1) shows that $\Delta_{\pm}^{(p)} \sim \gamma^{2p}/\epsilon^{2p-1}$ and, therefore, the expansion (1.1) becomes singular for $\epsilon = O(1/N)$, in agreement with our expectations. The relation (2.9) takes into account an infinite class of the leading corrections $O(1/(\epsilon^{2p-1}N^{2p}))$ to all orders in $1/N$ and it remains finite for $\epsilon \rightarrow 0$. At the crossing point, for $\epsilon = 0$, we find from (2.9) that $\Delta_{\pm} = \Delta_1 \pm \gamma/N$, so that the leading nonplanar correction to the resummed scaling dimensions scales as $O(1/N)$ (and not $O(1/N^2)$ as one would expect from (1.1)).⁴

Let us assume that the scaling dimensions Δ_1 and Δ_2 are continuous functions of the coupling constant g , intersecting in the planar limit at $g = 0$. Then, the relation (2.9) defines two functions $\Delta_{\pm}(g)$ that satisfy the non-crossing rule. Namely, the two levels $\Delta_+(g)$ and $\Delta_-(g)$ approach each other for $g = 0$ but remain separated by a finite gap (see figure 1)

$$|\Delta_+ - \Delta_-| \geq \frac{2\gamma}{N}. \quad (2.12)$$

We recall that the parameter γ defines the leading nonplanar correction to the two-point correlation function (2.2) and (2.5). As can be seen from figure 1, the level crossing leads to a change of behavior of the scaling dimensions in the vicinity of the crossing point. Away from the crossing point, the resummed scaling dimensions approach their values in the planar limit, e.g. $\Delta_+ \approx \Delta_1$ at large negative g goes into $\Delta_+ \approx \Delta_2$ at large positive g .

3 Resummed OPE coefficients

In the previous section, we defined the conformal primary operators O_{\pm} in the transition region (1.2) and determined their scaling dimensions (2.9). Let us examine the properties

³I would like to thank Sergey Frolov for suggesting this interpretation.

⁴Analogous phenomenon has been previously observed for critical dimensions of composite operators in the nonlinear $O(N)$ sigma-model [3]. At large N their expansion runs in powers of $1/N$ but the leading nonplanar correction scales as $O(1/\sqrt{N})$ due to the level crossing.

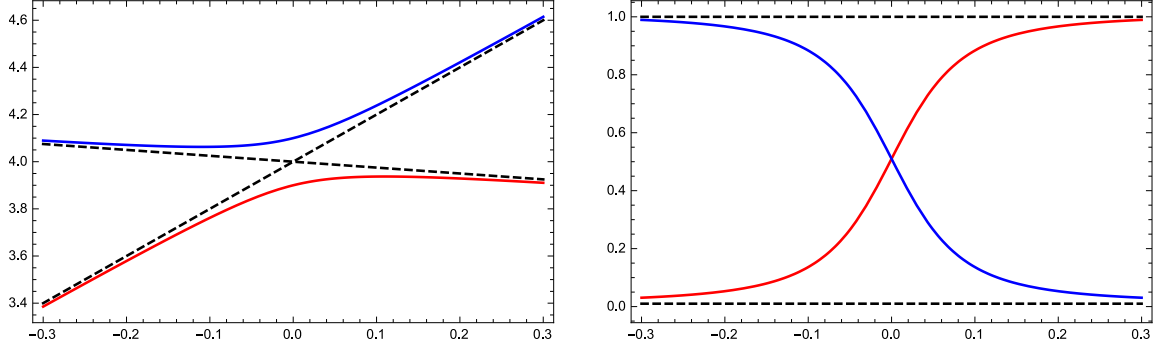


Figure 1. The scaling dimensions Δ_{\pm} (left panel) and the OPE coefficients $(C_{\phi\phi O_{\pm}}/C_{\phi\phi O_1})^2$ (right panel) in the transition region (1.2) as a function of the coupling constant g for $\gamma/N = 0.1$. The dashed lines represent the same quantities in the planar limit, $\Delta_i(g) = \alpha_i g + O(g^2)$.

of the OPE coefficients $C_{\phi\phi O_{\pm}}$ defining the correlation function $\langle\phi\phi O_{\pm}\rangle$ involving scalar conformal operator ϕ with the scaling dimension Δ_{ϕ} separated from Δ_{\pm} by a finite gap, $|\Delta_{\phi} - \Delta_{\pm}| = O(N^0)$.

Let us consider a four-point function of the operators ϕ and decompose it over the conformal blocks describing the contribution of different conformal multiplets

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_{\phi}}} \left[\mathcal{F}_{O_+}(u, v) + \mathcal{F}_{O_-}(u, v) + \sum_{\phi_i} \mathcal{F}_{\phi_i}(u, v) \right], \quad (3.1)$$

where $x_{ij} = x_i - x_j$, $u = x_{12}^2 x_{34}^2 / (x_{13}^2 x_{24}^2)$ and $v = x_{23}^2 x_{14}^2 / (x_{13}^2 x_{24}^2)$. For $x_{12} \rightarrow 0$, or equivalently $u \rightarrow 0$ and $v \rightarrow 1$, the contribution of the operators O_{\pm} to (3.1) takes the form

$$\mathcal{F}_{O_{\pm}}(u, v) = C_{\phi\phi O_{\pm}}^2 u^{\Delta_{\pm}/2} + \dots, \quad (3.2)$$

where dots denote terms suppressed by powers of u and $1 - v$.

The OPE coefficients in (3.2) are given by $C_{\phi\phi O_{\pm}}^2 \sim \langle\phi\phi O_{\pm}\rangle\langle O_{\pm}\phi\phi\rangle/\langle O_{\pm}O_{\pm}\rangle$. Replacing O_{\pm} with their explicit expressions (2.3), we find using (2.6)

$$\begin{aligned} C_{\phi\phi O_+}^2 &= \frac{(C_{\phi\phi O_1} - c_1 C_{\phi\phi O_2})^2}{1 + c_1^2}, \\ C_{\phi\phi O_-}^2 &= \frac{(C_{\phi\phi O_2} + c_1 C_{\phi\phi O_1})^2}{1 + c_1^2}, \end{aligned} \quad (3.3)$$

where $C_{\phi\phi O_i}$ are the OPE coefficients for the operators O_i in the planar limit and

$$c_1 = -\frac{N\epsilon}{2\gamma} + \sqrt{\frac{N^2\epsilon^2}{4\gamma^2} + 1} = \frac{\gamma}{N\epsilon} - \left(\frac{\gamma}{N\epsilon}\right)^3 + \dots, \quad (3.4)$$

for $\epsilon > 0$. The relation (3.3) defines two smooth functions of ϵ that cross each other for $\epsilon \neq 0$ and take the values $C_{\phi\phi O_{\pm}}^2 = (C_{\phi\phi O_1} \pm C_{\phi\phi O_2})^2/2$ at the crossing point $\epsilon = 0$.

The four-point correlation function (3.1) is a well-defined function of the parameters of the underlying CFT. In particular, at large N it admits an expansion that runs in powers

of $1/N^2$. It is natural to require that each term of this expansion should be finite for $\epsilon \rightarrow 0$. As we will see in the moment, this requirement leads to nontrivial consequences for the OPE coefficients $C_{\phi\phi O_i}$.

The conformal blocks entering the right-hand side of (3.1) depend on the scaling dimensions of the operators ϕ_i . Since Δ_{ϕ_i} are well-separated from each other in the transition region (1.2), the corresponding conformal blocks \mathcal{F}_{ϕ_i} automatically satisfy the regularity condition mentioned above. This is not the case however for the conformal blocks $\mathcal{F}_{O_{\pm}}$. According to (2.11) and (3.4), the large N expansion of Δ_{\pm} and c_1 is singular in the region (1.2) and, as a consequence, the coefficients of the expansion of (3.2) in powers of $1/N$ contain poles in ϵ . They cancel however in the sum $\mathcal{F}_{O_+} + \mathcal{F}_{O_-}$ that remains finite for $\epsilon \rightarrow 0$. Namely,

$$\mathcal{F}_{O_+} + \mathcal{F}_{O_-} = \left(C_{\phi\phi O_1}^2 u^{\epsilon/4} + C_{\phi\phi O_2}^2 u^{-\epsilon/4} \right) - 2\gamma C_{\phi\phi O_1} C_{\phi\phi O_2} \frac{u^{\epsilon/4} - u^{-\epsilon/4}}{N\epsilon} + O(1/N^2). \quad (3.5)$$

Notice that the second term on the right-hand side is finite for $\epsilon \rightarrow 0$ but it scales as $O(1/N)$.

It is straightforward to verify that the subleading corrections to (3.5) involve both even and odd powers of $1/N$. This seems to be in contradiction with the requirement that large N expansion of the correlation function (3.1) should run in even powers of $1/N$. We can eliminate all terms with odd powers of $1/N$ by imposing the additional condition on the OPE coefficients⁵

$$\frac{C_{\phi\phi O_2}}{C_{\phi\phi O_1}} = O(1/N). \quad (3.6)$$

This relation implies that if the scaling dimensions of two operators O_1 and O_2 approach each other and satisfy (1.2), their OPE coefficients in the planar limit have to differ by a large factor of N .

Combining together (3.6) and (3.3), we obtain the following expression for the resummed OPE coefficients

$$C_{\phi\phi O_{\pm}}^2 = \frac{1}{2} C_{\phi\phi O_1}^2 \left[1 \pm \frac{\epsilon}{\sqrt{\epsilon^2 + 4\gamma^2/N^2}} \right], \quad (3.7)$$

where $\epsilon = \Delta_1 - \Delta_2$ and $C_{\phi\phi O_1}$ are the OPE coefficients in the planar limit. Let us examine the expansion of (3.7) at large N and $\epsilon > 0$

$$\begin{aligned} C_{\phi\phi O_+}/C_{\phi\phi O_1} &= 1 - \frac{\gamma^2}{2N^2\epsilon^2} + O(1/N^4), \\ C_{\phi\phi O_-}/C_{\phi\phi O_1} &= \frac{\gamma}{N\epsilon} \left(1 - \frac{3\gamma^2}{2N^2\epsilon^2} + O(1/N^4) \right). \end{aligned} \quad (3.8)$$

Notice that the leading correction to $C_{\phi\phi O_-}$ contains a pole at $\epsilon = 0$. Its appearance is an artifact of the large N expansion. As follows from (3.7), the resummed expression for $C_{\phi\phi O_-}$ is finite and it takes the value $C_{\phi\phi O_1}/\sqrt{2}$ for $\epsilon = 0$. Away from the crossing point, for $|\epsilon N| \gg 1$, the structure constants $C_{\phi\phi O_{\pm}}$ coincide with those in the planar limit, $C_{\phi\phi O_1}$ and $C_{\phi\phi O_2}$.

The dependence of the OPE coefficients $(C_{\phi\phi O_{\pm}}/C_{\phi\phi O_1})^2$ on the coupling constant in the vicinity of the crossing point is shown in figure 1. We demonstrate in the next section

⁵There is obviously another possibility $C_{\phi\phi O_1}/C_{\phi\phi O_2} = O(1/N)$ but it leads to the same result for $C_{\phi\phi O_{\pm}}^2$.

that the relations (3.7) and (3.8) are in a good agreement with the known properties of three- and four-dimensional CFT's.

We recall that the relation (3.7) was obtained from the requirement for the four-point correlation function (3.1) to have a regular $1/N^2$ expansion as $\epsilon \rightarrow 0$. We substitute (3.7) into (3.5) and verify that this condition is indeed verified

$$\mathcal{F}_{O_+} + \mathcal{F}_{O_-} \sim u^{\Delta_1/2+\epsilon/4} \left\{ 1 + \frac{\gamma^2}{8N^2} \left[(\ln u)^2 - \frac{\epsilon}{6} (\ln u)^3 + O(\epsilon^2) \right] + O(1/N^4) \right\}. \quad (3.9)$$

A distinguished feature of this relation as compared with the analogous asymptotic behaviour of the conformal blocks is the appearance of terms enhanced by powers of $\ln u$. For $\epsilon = 0$ the $1/N$ -expansion on the right-hand side of (3.9) can be resummed leading to

$$\mathcal{F}_{O_+} + \mathcal{F}_{O_-} \sim \frac{1}{2} u^{(\Delta_1+\gamma/N)/2} + \frac{1}{2} u^{(\Delta_1-\gamma/N)/2}. \quad (3.10)$$

We recognize the two terms on the right-hand side as describing the leading $u \rightarrow 0$ asymptotics of conformal blocks corresponding to two conformal primary operators with the scaling dimensions $\Delta_1 \pm \gamma/N$.

4 Examples of level-crossing

In this section, we review the known examples of the level-crossing phenomenon in three- and four-dimensional CFT's and make a comparison with the results obtained in the previous sections.

4.1 Level-crossing in $\mathcal{N} = 4$ SYM

Our first example concerns the mixing of the Konishi and double-trace operators in four-dimensional $\mathcal{N} = 4$ SYM theory. As was mentioned in the Introduction, the scaling dimensions of these operators collide in the planar limit at finite value of the coupling constant.

In planar $\mathcal{N} = 4$ SYM, the Konishi and double-trace operators take the form

$$\begin{aligned} \mathcal{O}_K &= \frac{1}{N} \text{tr} [\Phi^I \Phi^I], \\ \mathcal{O}_D &= \frac{1}{N^J} \text{tr} [\Phi^{(I_1} \dots \Phi^{I_J)}] \text{tr} [\Phi^{(I_1} \dots \Phi^{I_J)}], \end{aligned} \quad (4.1)$$

where Φ^I (with $I = 1, \dots, 6$) are real scalar fields and $\text{tr}[\Phi^{(I_1} \dots \Phi^{I_J)}]$ is a symmetric traceless $\text{SO}(6)$ tensor. The normalization factors on the right-hand side of (4.1) were introduced for the two-point correlation functions of the operators to scale as $O(N^0)$ at large N .

At Born level, for zero coupling constant, the scaling dimensions of the operators (4.1) are $\Delta_K = 2$ and $\Delta_D = 2J$, respectively. At strong coupling, the scaling dimensions can be computed using the AdS/CFT correspondence, see e.g. [4, 5]

$$\begin{aligned} \Delta_D &= 2J - \frac{2(J-1)J(J+2)}{N^2} + \dots, \\ \Delta_K &= 2\lambda^{1/4} - 2 + \frac{2}{\lambda^{1/4}} + \dots, \end{aligned} \quad (4.2)$$

where dots denote terms suppressed by powers of $1/N^2$ and $\lambda^{-1/2}$. We observe that, at strong coupling, for $J \approx \lambda^{1/4}$ the two levels cross each other.

To establish the connection with the results of the previous sections, we examine three-point correlation functions of the operators (4.1) and half-BPS operators

$$O_J(x, Y) = \frac{1}{N^{J/2}} Y_{I_1} \dots Y_{I_J} \text{tr} \left[\Phi^{(I_1} \dots \Phi^{I_J)} \right], \quad (4.3)$$

where Y_I is an auxiliary six-dimensional null vector, $Y_I^2 = 0$. This operator carries the R -charge J and its scaling dimension is protected from quantum corrections, $\Delta_{O_J} = J$. The form of the three-point functions is fixed by conformal symmetry

$$\begin{aligned} \langle O_J(1) O_J(2) O_K(0) \rangle &= (Y_1 Y_2)^J \frac{C_{JJK}}{x_1^{\Delta_K} x_2^{\Delta_K} x_{12}^{2J-\Delta_K}}, \\ \langle O_J(1) O_J(2) O_D(0) \rangle &= (Y_1 Y_2)^J \frac{C_{JJD}}{x_1^{\Delta_D} x_2^{\Delta_D} x_{12}^{2J-\Delta_D}}. \end{aligned} \quad (4.4)$$

where $O_J(i) \equiv O_J(x_i, Y_i)$. The correlation function in the second line of (4.4) factors out in the planar limit into the product of two-point correlation functions of half-BPS operators (4.3) leading to

$$C_{JJD} = 1 + O(1/N^2). \quad (4.5)$$

The OPE coefficient C_{JJK} was computed at strong coupling in ref. [6]. It was found that C_{JJK} develops a pole at $J = \Delta_K/2 \approx \lambda^{1/4}$,⁶

$$C_{JJK} = \frac{J^{3/2} \sqrt{M}}{2N(J - \lambda^{1/4})}, \quad (4.6)$$

where $M = (J+1)(J+2)^2(J+3)/12 \approx J^4/12$ is the dimension of the J -symmetric traceless representation of the $SO(6)$. As was argued in [6], the appearance of a pole in (4.6) leads to the mixing of the Konishi operator O_K and the double-trace operator O_D .

Let us compare the relations (4.5) and (4.6) with the general expression (3.8). Upon identification of the operators, $\phi_i = O_J$, $O_+ = O_D$ and $O_- = O_K$, we find a perfect agreement with the leading term in (3.8) for

$$\epsilon = \Delta_D - \Delta_K \approx 2 \left(J - \lambda^{1/4} \right), \quad \gamma = \lambda^{3/8} \sqrt{M} \approx \frac{J^{7/2}}{2\sqrt{3}}. \quad (4.7)$$

Substituting these expressions into (3.8), we can determine the subleading terms in (4.5) and (4.6). Moreover, we can apply (2.9) and (3.7) to obtain the resummed expressions for the scaling dimensions and the OPE coefficients in the transition region $|J - \lambda^{1/4}| = O(1/N)$

$$\begin{aligned} \Delta_{\pm} &= J + \lambda^{1/4} \pm \sqrt{(J - \lambda^{1/4})^2 + \frac{J^7}{12N^2}}, \\ C_{JJO_{\pm}}^2 &= \frac{1}{2} \left[1 \pm \frac{J - \lambda^{1/4}}{\sqrt{(J - \lambda^{1/4})^2 + J^7/(12N^2)}} \right], \end{aligned} \quad (4.8)$$

⁶The three-point correlation functions in which the dimension of one operator equal to the sum of the other two are known as extremal correlators. The appearance of poles is a generic feature of the extremal correlators in the AdS/CFT correspondence [7–9].

where the subscripts $+/-$ correspond to the double-trace and Konishi operators, respectively. At the crossing point, for $J \approx \lambda^{1/4}$, the level splitting is

$$|\Delta_+ - \Delta_-| = \frac{J^{7/2}}{N\sqrt{3}}. \quad (4.9)$$

This expression is in agreement with the one given in ref. [6] (see the latest version). The structure constants (4.8) are finite in the transition region and do not require the additional renormalization of operators advocated in [6].

The dependence of Δ_{\pm} on the coupling constant in the vicinity of $\lambda^{1/4} \approx J$ follows the same pattern as shown in figure 1. In particular, examining Δ_- as a function of the coupling constant, we find that it grows for $\lambda^{1/4} \ll J$ as the scaling dimension of the Konishi operator Δ_K and, then, approaches the scaling dimension of the double-trace operator Δ_D for $\lambda^{1/4} \gg J$.⁷ As a consequence, it satisfies the relation $\Delta_- < \Delta_D < 2J$, so that the correlation function $\langle O_J(1)O_J(2)O_-(0) \rangle$ can not be extremal for finite N (see footnote 6).⁸

As the second example, we consider the mixing of leading-twist operators carrying a nonzero Lorentz spin S . The scaling dimension of twist-two operators grows at large spin as $\Delta_2 = 2 + S + \gamma_S$, where $\gamma_S = 2\Gamma_{\text{cusp}}(\lambda) \ln \bar{S} + O(S^0)$ with $\Gamma_{\text{cusp}}(\lambda)$ being the cusp anomalous dimension and $\bar{S} = S e^{\gamma_E}$ (with γ_E being the Euler constant) [12]. For twist-four operators, the lowest scaling dimension scales at large spin as $\Delta_4 = 4 + S - c(N)/S^2$ with positive $c(N)$ [13–15]. Comparing $\Delta_2(S)$ and $\Delta_4(S)$ we find that the two functions cross each other at $\gamma_S = 2 + O(1/S^2)$, or equivalently $\Gamma_{\text{cusp}}(\lambda) \sim 1/\ln \bar{S}$. The important difference with the previous example is that the crossing point is located at weak coupling and, therefore, it can be analyzed using perturbation theory.

The three-point correlation function of two half-BPS operators (4.3) with charge $J = 2$ and twist-two operator with large spin S has been computed at weak coupling in ref. [16] leading to the following result for the corresponding OPE coefficient

$$C_2(S)/C_2^{(0)}(S) = \frac{1}{N} \Gamma \left(1 - \frac{1}{2} \gamma_S(\lambda) \right) e^{-\frac{1}{2} \gamma_S(\lambda) [\gamma_E + \alpha(\lambda)] + \beta(\lambda)}, \quad (4.10)$$

where $C_2^{(0)}(S) = [2\Gamma^2(S + \gamma_S/2 + 1)/\Gamma(2S + \gamma_S + 1)]^{1/2}$ and functions $\alpha(\lambda)$ and $\beta(\lambda)$ are known explicitly up to two loops. The expression on the right-hand side of (4.10) was obtained for $S \gg 1$ with $\lambda \ln S = \text{fixed}$, it takes into account all corrections of the form $\lambda^k (\ln S)^n$ (with $1 \leq n \leq k$). For twist-four operators of the schematic form $O_2 \partial^S O_2$ the analogous OPE coefficient is

$$C_4(S)/C_4^{(0)}(S) = 1 + O(1/N^2), \quad (4.11)$$

where $C_4^{(0)}(S) = [(S+1)!(S+2)!/(2S+1)!]^{1/2}$.

⁷To the left of the crossing point, for $\lambda^{1/4} \ll J$, we have $O_+ = O_D$ and $O_- = O_K$, whereas to the right of the crossing point, for $\lambda^{1/4} \gg J$, the operators are exchanged $O_+ = O_K$ and $O_- = O_D$. At the crossing point, for $\lambda^{1/4} \approx J$, the operators are maximally mixed, $O_{\pm} = O_K \pm O_D$.

⁸The extremal situation can arise however for the correlation functions of protected operators $\langle O_{J_1}(1)O_{J_2}(2)O_{J_3}(3) \rangle$ for $J_3 = J_1 + J_2$. In this case, the level crossing does not occur since the scaling dimensions and the OPE coefficients are protected by $\mathcal{N} = 4$ superconformal symmetry. The calculation of such correlators in the AdS/CFT turns out to be a nontrivial task [8–11].

We observe that the twist-two OPE coefficient (4.10) develops a pole at $\gamma_S(\lambda) = 2$. Since $\Gamma_{\text{cusp}} = \lambda/(4\pi^2) + O(\lambda^2)$ at weak coupling, the corresponding value of the spin S is exponentially large, $S = \exp(4\pi^2/\lambda - \gamma_E + O(\lambda))$. Denoting $\epsilon = \Delta_4 - \Delta_2 = 2 - \gamma_S$, we find that for $\epsilon \rightarrow 0$ the relations (4.10) and (4.11) are in a perfect agreement with (3.8)

$$\frac{C_2(S)}{C_4(S)} = \frac{2e^{-\gamma_E}}{\epsilon NS} [1 + O(\lambda, 1/N^2)] . \quad (4.12)$$

The residue at $1/\epsilon$ pole defines the interaction energy γ between twist-two and twist-four operators

$$\gamma = \frac{2e^{-\gamma_E}}{S} = 2e^{-4\pi^2/\lambda + O(\lambda)} , \quad (4.13)$$

which is exponentially small at weak coupling. Applying (2.9) and (3.7), we can obtain the resummed expressions for the scaling dimensions of these operators and their OPE coefficients in the transition region $|\Delta_4 - \Delta_2| = O(1/N)$.

4.2 Level-crossing in three-dimensional CFT

Recently an important progress has been achieved in understanding the properties of three-dimensional unitary CFT's within the conformal bootstrap approach. It was found that the constraints of crossing symmetry and unitarity restrict the possible values of the scaling dimensions of the operators Δ_i in a highly nontrivial way [17, 18].

For our purposes we shall examine the extremal solutions to these constraints living on the boundary of the allowed region of Δ_i 's. The spectrum of the corresponding three-dimensional unitary CFT's can be uniquely reconstructed and it is parameterized by the dimension Δ_σ of the leading \mathbb{Z}_2 -odd scalar operator σ . As was shown in [18], the resulting scaling dimensions reveal a dramatic transition at $\Delta_\sigma^* = 0.518154(15)$. The values of Δ_i 's at the transition point are remarkably close to the expected values of the critical exponents of three-dimensional Ising model.

Another intriguing feature of the 3d Ising point is that some operators disappear from the spectrum. As an example, consider the leading spin-2 operators that appears in the OPE $\sigma \times \sigma$. The lowest lying operator is the stress-energy tensor T whose conformal weight $\Delta_T = 3$ is protected and is independent on Δ_σ . For the next-to-lowest spin-2 operator T' , the scaling dimension and the OPE coefficient at the 3d Ising point are given by $\Delta_{T'}^* = 5.500(15)$ and $(C_{\sigma\sigma T'}^*)^2 = 2.97(2) \times 10^{-4}$, respectively. However, slightly above this point, for $\Delta_\sigma = 0.5182$, the OPE coefficient decreases by the order of magnitude $C_{\sigma\sigma T'}^2 = 2.21 \times 10^{-5}$ and the scaling dimension changes to $\Delta_{T'} = 4.334$ (see figure 16 in ref. [18]).

A close examination of the spectrum of lowest spin 2 operators (see figure 15 in ref. [18]) shows that in the vicinity of the 3d Ising point the levels approach each other in a pairwise manner. Moreover, the variation of the scaling dimensions and the OPE coefficients in the transition region follows the same pattern as the one shown in figure 1. This suggests that the spectrum in this region can be described by the relations (2.9) and (3.3). Important difference with the previous case is that the three-dimensional CFT's under consideration do not have a natural parameter like $1/N$ and the arguments leading to (3.6) and (3.7) do

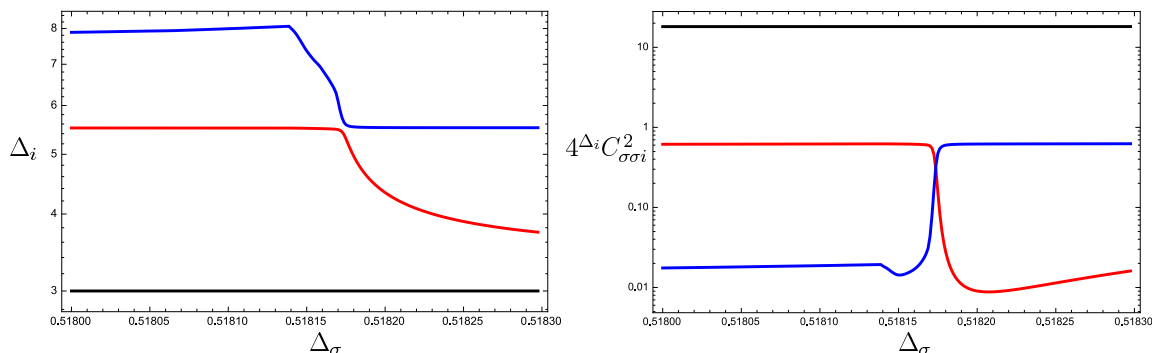


Figure 2. The scaling dimensions of the three lowest spin-2 operators and their OPE coefficients (multiplied by 4^{Δ_i}) as found in ref. [18].

not apply. In what follows we shall apply (2.9) and (3.3) and interpret the parameter γ/N as ‘interaction energy’ v between levels.

Let us examine the Δ_σ -dependence of the scaling dimensions and the OPE coefficients of three lowest levels shown in figure 2. As was already mentioned, the lowest state with $\Delta_T = 3$ (black line) corresponds to the stress-energy tensor. For the next-to-lowest level (red line), the scaling dimension Δ_- and the OPE coefficient $C_{\sigma\sigma-}$ vary slowly with Δ_σ to the left from the 3d Ising point, for $\Delta_\sigma < \Delta_\sigma^*$, and approach the following values for $\Delta_\sigma \ll \Delta_\sigma^*$

$$\Delta_1 = 5.512, \quad C_{\sigma\sigma 1}^2 = 2.984 \times 10^{-4}. \quad (4.14)$$

For the next-to-next-to-lowest level (blue line), Δ_+ and $C_{\sigma\sigma+}$ vary slowly with Δ_σ to the right from the 3d Ising point, for $\Delta_\sigma > \Delta_\sigma^*$, and approach the same values (4.14) for $\Delta_\sigma \gg \Delta_\sigma^*$.

To describe the variation of Δ_\pm and $C_{\sigma\sigma\pm}$ in the transition region, for

$$0.51815 < \Delta_\sigma < 0.51820, \quad (4.15)$$

we introduce the level splitting function $\epsilon = \epsilon(\Delta_\sigma)$ analogous to (1.2)

$$\epsilon = 2\Delta_1 - \Delta_- - \Delta_+. \quad (4.16)$$

It is straightforward to verify that away from the transition region ϵ measures the difference between the scaling dimensions: $\epsilon = \Delta_- - \Delta_+ < 0$ for $\Delta_\sigma \ll \Delta_\sigma^*$ and $\epsilon = \Delta_+ - \Delta_- < 0$ for $\Delta_\sigma \gg \Delta_\sigma^*$. We find that ϵ is a smooth monotonic function of Δ_σ in the transition region (4.15) and it vanishes for $\Delta_\sigma = 0.51817$. Then, we can invert the function $\epsilon = \epsilon(\Delta_\sigma)$ and obtain the dependence of Δ_\pm and $C_{\sigma\sigma\pm}$ on ϵ as shown in figure 3.

Applying (2.9), we expect that the scaling dimensions $\Delta_\pm(\epsilon)$ in the transition region (4.15) are given by

$$\Delta_\pm(\epsilon) = \Delta_1 - \frac{\epsilon}{2} \pm \sqrt{\frac{\epsilon^2}{4} + v^2}, \quad (4.17)$$

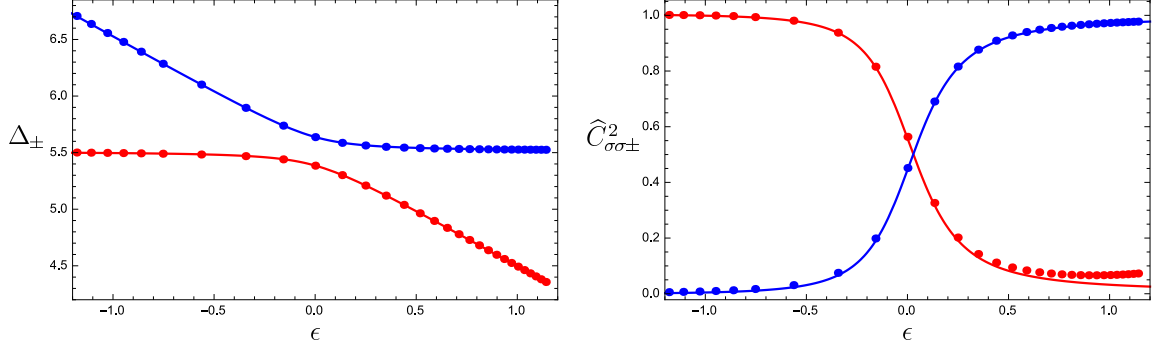


Figure 3. Dependence of the scaling dimensions of the spin-2 operators and their OPE coefficients $\hat{C}^2_{\sigma\sigma\pm} = C^2_{\sigma\sigma\pm}/C^2_{\sigma\sigma 1}$ on the level splitting ϵ . Dots stand for the exact values found in [18], solid lines are described by (4.17) and (4.18).

where Δ_1 is defined in (4.14) and v is the interaction energy. For the OPE coefficients, we apply (3.3) and (3.4) to get

$$\begin{aligned} C^2_{\sigma\sigma+}(\epsilon) &= \frac{[C_{\sigma\sigma 1} - c_1 C_{\sigma\sigma 2}]^2}{1 + c_1^2}, \\ C^2_{\sigma\sigma-}(\epsilon) &= \frac{[C_{\sigma\sigma 2} + c_1 C_{\sigma\sigma 1}]^2}{1 + c_1^2}, \end{aligned} \quad (4.18)$$

where $c_1(\epsilon) = (\sqrt{\epsilon^2 + 4v^2} - \epsilon)/(2v)$ and $C_{\sigma\sigma 1}$ is given by (4.14). To fix v and $C_{\sigma\sigma 2}$, we compare (4.17) and (4.18) with the results of ref. [18] shown by dots in figure 3. We find a good agreement for

$$v = 0.126, \quad C_{\sigma\sigma 2}/C_{\sigma\sigma 1} = 0.057. \quad (4.19)$$

These values depend on the number of the crossing-symmetry constraints $n_{\text{comp}} = 231$ used in ref. [18] and they are expected to decrease as $n_{\text{comp}} \rightarrow \infty$. An immediate consequence of (4.18) and (4.19) is that the OPE coefficient $C^2_{\sigma\sigma-}(\epsilon)$ decreases by the order of magnitude in the transition region (4.15). This result is in a quantitative agreement with the expected properties of the operator T' mentioned in the beginning of the subsection.

We observe from figure 3 that the OPE coefficients $C^2_{\sigma\sigma-}$ given by (4.18) start to deviate from their exact values for $\epsilon > 0.5$. We recall that the relations (4.18) were obtained under the assumption that the levels Δ_{\pm} are separated from the rest of the spectrum of Δ_i by a finite gap, e.g. $|\Delta_+ - \Delta_-| \ll |\Delta_- - \Delta_T|$. Since $\Delta_-(\epsilon)$ decreases linearly with ϵ , this condition is not fulfilled for $\epsilon \gg v$ and, as a consequence, $C^2_{\sigma\sigma-}$ receives the additional corrections due to mixing of the operator O_- with the stress-energy tensor.

It is interesting to note that, according to (4.19), the ratio of the OPE coefficients $C_{\sigma\sigma 2}/C_{\sigma\sigma 1}$ is anomalously small. This makes the properties of three-dimensional CFT's in the vicinity of critical 3d Ising model similar to those of four-dimensional $\mathcal{N} = 4$ SYM theory in which case the OPE coefficients satisfy (3.6) at large N . Notice that for the operators with sufficiently large scaling dimensions in unitary CFT's, their OPE coefficients

have to fall off exponentially fast [19]. Yet another surprising feature of 3d Ising point is that the same asymptotic behaviour also holds for the lowest spin-2 operators.

We would like to emphasize that the above analysis relies on the assumption that the extremal solutions living on the boundary of the allowed region of Δ_i 's admit an interpretation in terms of three-dimensional CFT's, so that the curves shown in figure 2 can be interpreted as describing a continuous flow of the conformal data.⁹ This assumption is non-trivial and it does not hold e.g. for two-dimensional CFT's. In the latter case, the spectrum of the scaling dimensions of the extremal solutions to the left of the two-dimensional Ising point is inconsistent with Virasoro symmetry making a CFT interpretation impossible [18].

4.3 Level-crossing in QCD

As another example of level-crossing phenomenon, we consider the spectrum of scaling dimensions of composite operators that appeared in the study of the QCD evolution equations for three-particle distribution amplitudes [20, 21]. In the simplest case of baryon distribution amplitude of helicity 1/2, these operators involve covariant derivatives acting on three quark fields and have the following form

$$B_{n_1 n_2 n_3} = \epsilon^{ijk} D^{n_1} q_i^\uparrow D^{n_2} q_j^\downarrow D^{n_3} q_k^\uparrow, \quad S = n_1 + n_2 + n_3, \quad (4.20)$$

where $q_i^{\uparrow(\downarrow)} = \frac{1}{2}(1 \pm \gamma_5)q_i$ are quark fields with definite chirality and with color $i = 1, 2, 3$ corresponding to the fundamental representation of the SU(3) gauge group.

As was shown in [20], the one-loop dilatation operator acting on the operators $B_{n_1 n_2 n_3}$ can be mapped into a Hamiltonian of a spin chain of length 3

$$\mathbb{H}_{1/2} = H_1 + \epsilon H_2, \quad (4.21)$$

where spin operators at each site are identified with the generators of the collinear SL(2) subgroup of the conformal group¹⁰ acting on quark fields in (4.20). Here H_1 coincides with a Hamiltonian of a completely integrable Heisenberg SL(2) spin chain and the operator H_2 is given by

$$H_2 = -\left(\frac{1}{S_{12}^2} + \frac{1}{S_{23}^2}\right), \quad (4.22)$$

where S_{ij}^2 stands for the sum of two SL(2) spins at sites i and j . The eigenvalues of (4.21) define the spectrum of anomalous dimensions of baryonic operators (4.20) at one loop.

The Hamiltonian (4.21) depends on the parameter ϵ . For baryonic operators (4.20), its value is uniquely fixed $\epsilon = 1$. To understand the properties of (4.21), it is convenient to treat ϵ as a new coupling constant and examine the dependence of the eigenvalues of (4.21) on ϵ . For $\epsilon = 0$ the Hamiltonian (4.21) is completely integrable and its exact eigenspectrum can be found with a help of Bethe ansatz technique. For $\epsilon \neq 0$ the Hamiltonian (4.21) is neither integrable, nor cyclic symmetric. Nevertheless, its eigenspectrum can be determined for arbitrary ϵ using the technique described in [20, 21].

⁹I would like to thank Slava Rychkov for discussing this point.

¹⁰Although the conformal symmetry of QCD is broken at quantum level, the one-loop dilatation operator respects the conformal symmetry, see e.g. [22].

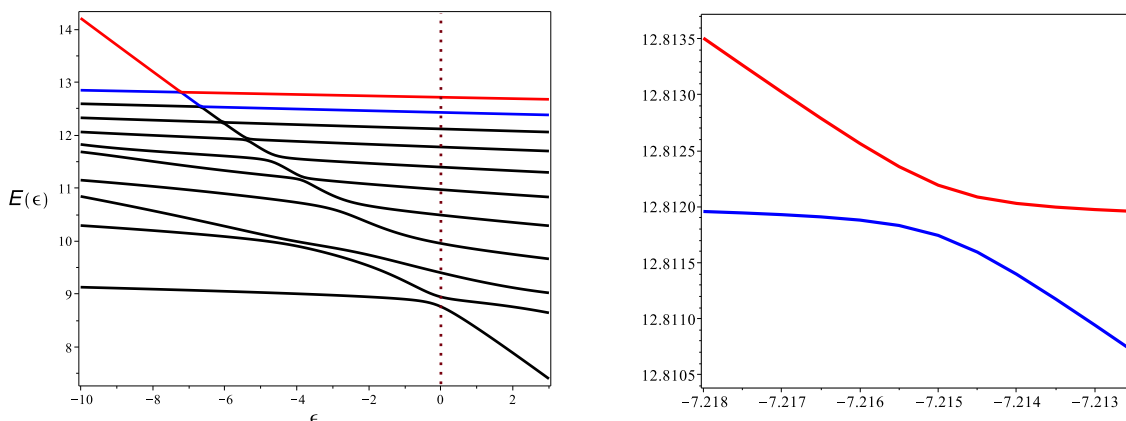


Figure 4. The flow of energies of parity-even eigenstates of the Hamiltonian (4.21) for $S = 20$. The vertical dotted line indicates the spectrum of H_1 . The two highest levels are shown by red and blue lines. The flow of energies close to the crossing point is zoomed in on the right panel.

The Hamiltonian (4.21) is invariant under the exchange of two sites $1 \leftrightarrow 3$ and its eigenstates can be classified according to the parity under this transformation. For $\epsilon = 0$ the eigenstates with different parity have the same energy in virtue of integrability, but the degeneracy is lifted for $\epsilon \neq 0$. The flow with ϵ of energy levels with different parity is independent from one another and the non-crossing rule is not applicable. At the same time, the levels with the same parity are not allowed to cross.

As an example, we show on figure 4 the ϵ -dependence of the energies of parity-even states with the total spin $S = 20$. We observe that for sufficiently large negative ϵ the energy levels approach each other. As was shown in [20], in the upper part of the spectrum, the crossing points correspond to the collision of levels of the Hamiltonians H_1 and H_2

$$H_1|\psi_1\rangle = e_1|\psi_1\rangle, \quad H_2|\psi_2\rangle = e_2|\psi_2\rangle, \quad (4.23)$$

with $\langle\psi_1|\psi_2\rangle \neq 0$ and $\langle\psi_i|\psi_i\rangle = 1$. Assuming that the rest of eigenstates is irrelevant in the vicinity of the crossing point, we can look for the eigenstate of (4.21) on a linear space spanned by the states $|\psi_1\rangle$ and $|\psi_2\rangle$. Then, it is convenient to define the state

$$|\hat{\psi}_1\rangle = c(|\psi_1\rangle - |\psi_2\rangle\langle\psi_2|\psi_1\rangle), \quad c = (1 - |\langle\psi_1|\psi_2\rangle|^2)^{-1/2}, \quad (4.24)$$

such that $\langle\hat{\psi}_1|\psi_2\rangle = 0$ and $\langle\hat{\psi}_1|\hat{\psi}_1\rangle = 1$, and evaluate the matrix elements of $\mathbb{H}_{1/2}$ with respect to $|\hat{\psi}_1\rangle$ and $|\psi_2\rangle$. In the standard manner, the diagonal matrix elements define corrections to the energies of two states

$$\begin{aligned} E_1(\epsilon) &= e_1 + \epsilon\langle\psi_1|H_2|\psi_1\rangle + O(\langle\psi_1|\psi_2\rangle^2), \\ E_2(\epsilon) &= \epsilon e_2 + \langle\psi_2|H_1|\psi_2\rangle. \end{aligned} \quad (4.25)$$

The off-diagonal matrix element defines their interaction energy

$$v = \langle\hat{\psi}_1|\mathbb{H}_{1/2}|\psi_2\rangle = c\langle\psi_1|\psi_2\rangle [e_1 - \langle\psi_2|H_1|\psi_2\rangle]. \quad (4.26)$$

It is proportional to the overlap of the two eigenstates $\langle \psi_1 | \psi_2 \rangle$ and does not depend on ϵ . As follows from (4.25), the levels could cross at $E_1(\epsilon) = E_2(\epsilon)$, or equivalently

$$\epsilon_\star = \frac{e_1 - \langle \psi_2 | H_1 | \psi_2 \rangle}{e_2 - \langle \psi_1 | H_2 | \psi_1 \rangle}. \quad (4.27)$$

The crossing does not happen due to nonzero interaction energy $|v| \neq 0$. The eigenvalues of $\mathbb{H}_{1/2}$ in the vicinity of (4.27) are given by

$$E_\pm(\epsilon) = \frac{1}{2}(E_1(\epsilon) + E_2(\epsilon)) \pm \sqrt{\frac{1}{4}(E_1(\epsilon) - E_2(\epsilon))^2 + |v|^2}. \quad (4.28)$$

To make this expression more explicit, we consider the limit of large spin $S \gg 1$. In this limit, $E_\pm(\epsilon)$ describe the one-loop anomalous dimensions of baryonic operators (4.20) with large number of derivatives S .

The Schrödinger equations (4.23) have been thoroughly studied in [20]. For the eigenstates with large total spin S , their energies can be expanded in powers of $1/S$ leading to¹¹

$$\begin{aligned} e_1 &= 6 \ln S - 3 \ln 3 - 6\psi(2) + \frac{6}{S}(2 - \ell_1) + O(1/S^2), \\ e_2 &= -\frac{1}{(\ell_2 + 1)(\ell_2 + 2)} + O(1/S^2), \end{aligned} \quad (4.29)$$

where $\psi(x) = (\ln \Gamma(x))'$ is Euler's digamma function and nonnegative integers ℓ_1 and ℓ_2 enumerate the energy levels. The eigenstates with the maximal $|e_i|$ correspond to $\ell_1 = \ell_2 = 0$. In the similar manner, for the matrix elements of Hamiltonians with respect to $|\psi_1(\ell_1)\rangle$ and $|\psi_2(\ell_2)\rangle$ we have

$$\begin{aligned} \langle \psi_1 | H_2 | \psi_1 \rangle &= O(1/S^2), \\ \langle \psi_2 | H_1 | \psi_2 \rangle &= 4 \ln S + C(\ell_2) + O(1/S), \end{aligned} \quad (4.30)$$

where $C(\ell_2) = -6\psi(2) + 6\psi(\ell_2 + 2) - 4\psi(2\ell_2 + 4) + 2/(\ell_2 + 2) + 2/(2\ell_2 + 3)$. Following [20], we can also compute the overlap of the eigenstates at large S

$$|\langle \psi_1 | \psi_2 \rangle| = \kappa S^{\ell_1/2 + 1/4} e^{-S/2}, \quad (4.31)$$

where $\kappa = e^{\ell_2 \sqrt{3}} [6\sqrt{\pi} \ell_1! (\ell_2 + 1)(\ell_2 + 2)/(2^{\ell_1} (2\ell_2 + 3))]^{-1/2}$.

Substituting the above relations into (4.26) and (4.27) we obtain the following expressions for the interaction energy

$$|v| = 2\kappa S^{\ell_1/2 + 1/4} e^{-S/2} \ln S, \quad (4.32)$$

and for the location of the crossing point

$$\epsilon_\star = -(\ell_2 + 1)(\ell_2 + 2) \left[2 \ln(S + 1) + c_0(\ell_2) - \frac{6\ell_1 + c_1(\ell_2)}{S} \right] + O(1/S^2), \quad (4.33)$$

¹¹Expression for e_1 in this relation only holds in the upper part of the spectrum close to the maximal energy level, the general expression for e_1 can be found in [20].

where $c_0 = -3 \ln 3 - 6\psi(2) - C(\ell_2)$ and c_1 is independent on ℓ_1 . We recall that the integers ℓ_1 and ℓ_2 enumerate the colliding levels.

We are now ready to compare the relations (4.28), (4.32) and (4.33) with the exact energy spectrum of the Hamiltonian (4.21) shown in figure 4 for $S = 20$. We remind that for $\epsilon = 0$ we have $\mathbb{H}_{1/2} = H_1$ and the energy levels are given by e_1 , eq. (4.29), for $\ell_1 = 0, 1, \dots$ starting from the maximal one. For $\epsilon \neq 0$ away from the crossing point, we find from (4.25) and (4.30) that $E_1(\epsilon) = e_1 + O(1/S^2)$, so that the variation of the energy with ϵ is very slow at large S . We observe from figure 4 that at large negative ϵ the maximal energy grows linearly with $(-\epsilon)$. According to (4.25), (4.29) and (4.30), it is given by

$$E_2(\epsilon) = -\frac{\epsilon}{(\ell_2 + 1)(\ell_2 + 2)} + 4 \ln S + C(\ell_2) + O(1/S), \quad (4.34)$$

for $\ell_2 = 0$. Going to smaller $(-\epsilon)$ we find that this level crosses subsequently the levels $E_1(\ell_1)$ for $\ell_1 = 0, 1, 2, \dots$. According to (4.33), the position of the crossing point scales as $\epsilon_\star(\ell_1) = -4 \ln S + 12\ell_1/S + \dots$, so that the distance between two subsequent points scales as $\epsilon_\star(\ell_1 + 1) - \epsilon_\star(\ell_1) = 12/S$. The interaction energy (4.32) defines the minimal distance between the colliding levels, $E_+(\epsilon_\star) - E_-(\epsilon_\star) = 2|v|$. As follows from (4.32), this distance is exponentially small at large spin S and increases as S^{ℓ_1} with the level number.

These results are in a good agreement with properties of the exact spectrum shown in figure 4. As an example, we consider the flow of two highest levels shown by red and blue lines in figure 4. The energies of these levels follow (4.28). They approach each other for $\epsilon_\star = -7.215$ and the level splitting is $2|v| = 4.46 \times 10^{-4}$. Applying (4.32) and (4.33) for $\ell_1 = \ell_2 = 0$ and $S = 20$, we find $\epsilon_\star = -7.229$ and $2|v| = 4.32 \times 10^{-4}$ which are close to the exact values.

Notice that the flow of energy levels in the lower part of the spectrum in figure 4 is more complicated and it deviates from (4.28). We recall that the relation (4.28) was derived under the assumption that the two levels E_1 and E_2 mix weakly with the rest of the spectrum. As was shown in [20], this assumption is not justified in the lower part of the spectrum due to a strong interaction between the energy levels. Although the lowest levels can be discussed avoiding the admixture of the higher ones, finding their energies amounts to diagonalization of the mixing matrix whose size grows for large spin as $\ln S$. The relation (4.28) does not apply in this case and more refined analysis is needed. The problem was solved in [20, 21] where it was shown that the effect of strong mixing is that the lowest state becomes separated from the rest by a finite gap.

5 Concluding remarks

In this note, we examined the properties of conformal operators whose scaling dimensions approach each other for some values of the parameters of a unitary CFT and satisfy von Neumann-Wigner non-crossing rule. We argued that the scaling dimensions of such operators and their OPE coefficients have a universal scaling behavior in the vicinity of the crossing point. We demonstrate that the obtained relations are in a good agreement with the known examples of the level-crossing phenomenon in maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory, three-dimensional conformal field theories and QCD.

In gauge theories, the scaling dimensions do not verify the non-crossing rule in the planar limit, but the level crossing leads to breakdown of $1/N$ expansion. Namely, the nonplanar corrections develop singularities close to the crossing point that have to be resummed to all orders in $1/N$. We argued that their resummation is very similar to resolving the mixing of operators in gauge theories at weak coupling with $1/N$ playing the role of the coupling constant. The resulting expressions for the scaling dimensions and the OPE coefficients, eqs. (2.9) and (3.7), respectively, remain finite in the transition region and obey the non-crossing rule. Away from this region their values coincide with those in the planar limit.

The scaling dimensions of twist-two operators are known to be increasing functions of the Lorentz spin S . For sufficiently large values of the coupling constant and/or the spin, they cross the scaling dimensions of the leading twist-four double-trace operators. For small (large) spin S , the crossing happens at strong (weak) coupling. We used the known results for Konishi operators ($S = 0$) and for twist-two operators with large spin ($S \gg 1$) to show that, due to the level crossing, the scaling dimensions and the OPE coefficients of twist-two and twist-four operators get swapped at the crossing point leading to a dramatic change of their asymptotic behaviour.¹²

This effect is believed to play an important role in verifying the conjectured S -duality in $\mathcal{N} = 4$ SYM [23, 24]. In particular, for finite N , the spectrum of the scaling dimensions should be invariant under the weak/strong coupling duality transformation $h \rightarrow 1/h$ with $h = g^2/(4\pi)$. In the special case of Konishi operator, the S -duality implies the existence of a nonperturbative operator K' whose scaling dimension at weak coupling is related to that of the Konishi operator as $\Delta_{K'}(h) = \Delta_K(1/h)$ [25]. Let us examine the flow of the scaling dimensions $\Delta_{K'}(h)$ and $\Delta_K(h)$ and ignore, for the sake of simplicity, the mixing with other operators. Obviously, the functions $\Delta_{K'}(h)$ and $\Delta_K(h)$ could cross each other at $h = 1$. As before, this can not happen due to a mixing between the operators K and K' . Replacing $\Delta_1 = \Delta_K(h)$ and $\Delta_2 = \Delta_K(1/h)$ in (2.9), we obtain the expressions for $\Delta_{\pm}(h)$ which verify the non-crossing rule and are invariant under the S -duality transformation, $\Delta_{\pm}(h) = \Delta_{\pm}(1/h)$. Adding more operators does not change this result but makes the corresponding picture of levels flow more complicated due to appearance of the additional crossing points.¹³ It would be interesting to test this mechanism using a lattice formulation of $\mathcal{N} = 4$ SYM [26].

Acknowledgments

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¹²Classification of the operators according to their twist becomes redundant at the crossing point, since the anomalous dimension of twist-two operator approaches the value $\gamma_S = 2$ and its twist increases to 4.

¹³Similar to the Konishi operator, each operator O_i is accompanied by a nonperturbative S -dual operator O'_i . Then, resolving the mixing of the operators at the crossing points, we find that their scaling dimensions are obtained by sewing together pieces of functions $\Delta_i(h)$ and $\Delta_i(1/h)$. The resulting functions are indeed invariant under the weak/strong coupling duality.

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